

Yang-Mills moduli space in the adiabatic limit

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Abstract

We consider the Yang-Mills equations for a matrix gauge group G inside the future light cone of 4-dimensional Minkowski space, which can be viewed as a Lorentzian cone $C(H^3)$ over the 3-dimensional hyperbolic space H^3 . Using the conformal equivalence of $C(H^3)$ and the cylinder $\mathbb{R} \times H^3$, we show that, in the adiabatic limit when the metric on H^3 is scaled down, classical Yang-Mills dynamics is described by geodesic motion in the infinite-dimensional group manifold $C^\infty(S_\infty^2, G)$ of smooth maps from the boundary 2-sphere $S_\infty^2 = \partial H^3$ into the gauge group G .

1. Yang-Mills theory with Higgs fields governs three fundamental forces of Nature. It has a number of particle-like solutions such as vortices, monopoles and instantons [1, 2, 3]. One may ask about the dynamics of vortices and monopoles which evolve according to the second-order field equations of Yang-Mills-Higgs theory. In the seminal paper [4] Manton suggested that in the “slow-motion limit” monopole dynamics can be described by geodesics in the moduli space of static multi-monopole solutions.¹ This approach was extended both to vortices in 2+1 dimensions (see e.g. [6] for a review) and instantons in 4+1 dimensions (see e.g. [7, 8]). In contrast, almost nothing is known about time-dependent solutions of pure Yang-Mills theory in 3+1 dimensions. Here we aim to partially fill this gap by applying Manton’s approach to the Yang-Mills equations on Minkowski space.

2. We parametrize Minkowski space-time $\mathbb{R}^{3,1}$ with coordinates x^μ , $\mu = 0, 1, 2, 3$, and the metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad (\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1) . \quad (1)$$

In this article we fix an origin in $\mathbb{R}^{3,1}$ and consider the time evolution of Yang-Mills fields in the interior of its light cone. For simplicity we will restrict ourselves to the future light cone L_+ and its interior T_+ only, as the considerations for the past are similar. L_+ and T_+ are defined by

$$(\tau^2 = 0, x^0 > 0) \quad \text{and} \quad (\tau^2 > 0, x^0 > 0) \quad \text{for} \quad \tau^2 = -\eta_{\mu\nu} x^\mu x^\nu , \quad (2)$$

respectively.

On T_+ one can introduce global pseudospherical coordinates $(\tau, \chi, \theta, \varphi)$ by

$$x^0 = \tau \cosh \chi, \quad x^1 = \tau \sinh \chi \sin \theta \cos \varphi, \quad x^2 = \tau \sinh \chi \sin \theta \sin \varphi, \quad x^3 = \tau \sinh \chi \cos \theta \quad (3)$$

and a range of

$$\tau \in (0, \infty), \quad \chi \in [0, \infty), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi) \quad (4)$$

with the usual identifications and a harmless coordinate singularity at $\chi = 0$. The eigentime coordinate τ foliates T_+ into a family of hyperbolic 3-spaces $H^3(\tau)$ or ‘radius’ τ , each of which is built from spheres $S^2(\chi)$ of radius $\tau \sinh \chi$. In these coordinates, the metric (1) acquires the form

$$ds^2 = -d\tau^2 + \tau^2 \{d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)\} , \quad (5)$$

where the expression in the round brackets is the metric on S^2 and the expression in the curly brackets is the metric on H^3 . For any given τ , the boundary $\partial H^3(\tau)$ is reached in the limit $\chi \rightarrow \infty$ and forms a 2-sphere S_∞^2 ‘at infinity’.

The metric (5) can be rewritten as

$$ds^2 = -d\tau^2 + \tau^2 \delta_{ab} e^a \otimes e^b = \tau^2 (-(\tau^{-1} d\tau)^2 + \delta_{ab} e^a \otimes e^b) , \quad (6)$$

where $\{e^a\}$ is a basis of one-forms on H^3 easily extracted from (5). From (6) we recognize a cone over H^3 , i.e. $T_+ = C(H^3)$, which is conformally equivalent to a cylinder $\mathbb{R} \times H^3$ with the metric

$$ds_{cyl}^2 = -du^2 + \delta_{ab} e^a \otimes e^b \quad \text{for} \quad u = \ln \tau \quad (7)$$

¹For nice reviews and a lot of references see e.g. [2, 3, 5].

and $H^3 = H^3(\tau=1)$. We redenote the cylindrical coordinates,

$$(u, \chi, \theta, \varphi) = (y^0, y^1, y^2, y^3) = (y^0, y^a) \quad \text{with } a = 1, 2, 3. \quad (8)$$

From this point on we will work on the cylinder (7) since Yang-Mills theory is conformally invariant.

3. We have set the stage to consider pure Yang-Mills theory on the cylinder $\mathbb{R} \times H^3$ with an arbitrary matrix gauge group G . The Yang-Mills potential $\mathcal{A} = \mathcal{A}_\mu dy^\mu$ takes its value in the Lie algebra $\mathfrak{g} = \text{Lie } G$ carrying a scalar product defined by the matrix trace Tr . The field tensor $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is defined as

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} dy^\mu \wedge dy^\nu \quad \text{with} \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu], \quad (9)$$

and the Yang-Mills equations read

$$D_\mu \mathcal{F}^{\mu\nu} := \sqrt{|\det g|}^{-1} \partial_\mu (\sqrt{|\det g|} \mathcal{F}^{\mu\nu}) + [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0, \quad (10)$$

where $g = (g_{\mu\nu})$ is the metric (7) on $\mathbb{R} \times H^3$.

For the metric (7) we have

$$\mathcal{A} = \mathcal{A}_0 dy^0 + \mathcal{A}_a dy^a = \mathcal{A}_0 dy^0 + \mathcal{A}_{H^3}, \quad (11)$$

$$\mathcal{F} = \mathcal{F}_{0a} dy^0 \wedge dy^a + \frac{1}{2} \mathcal{F}_{ab} dy^a \wedge dy^b = \mathcal{F}_{0a} dy^0 \wedge dy^a + \mathcal{F}_{H^3}. \quad (12)$$

Employing the adiabatic approach [4], we deform the metric (7) and introduce

$$ds_\varepsilon^2 = -du^2 + \varepsilon^2 \delta_{ab} e^a \otimes e^b, \quad (13)$$

where ε is a real positive parameter. Then $|\det g_\varepsilon| = \varepsilon^6 |\det g|$,

$$\mathcal{F}_\varepsilon^{0a} = g_\varepsilon^{00} g_\varepsilon^{ab} \mathcal{F}_{0b} = \varepsilon^{-2} \mathcal{F}^{0b} \quad \text{and} \quad \mathcal{F}_\varepsilon^{ab} = \varepsilon^{-4} \mathcal{F}^{ab}, \quad (14)$$

where in \mathcal{F}^{0a} and \mathcal{F}^{ab} the indices were raised by the non-deformed metric.

The adiabatic limit of scaling down the metric on H^3 is effected by the limit $\varepsilon \rightarrow 0$. To avoid the ε^{-1} divergence of the Yang-Mills action functional, one has to impose the vanishing of the curvature (12) along H^3 ,

$$\mathcal{F}_{H^3} = 0, \quad (15)$$

which renders the connection \mathcal{A}_{H^3} flat. Substituting (14) into the Yang-Mills equations on the cylinder $\mathbb{R} \times H^3$ with the metric (13) and taking the adiabatic limit $\varepsilon \rightarrow 0$ (corresponding to ‘slow u evolution’) together with $\mathcal{F}_{H^3} = 0$, we obtain

$$g^{ab} D_a \mathcal{F}_{b0} = 0, \quad (16)$$

$$D_0 \mathcal{F}_{0b} = 0, \quad (17)$$

which are, in fact, valid for any $\varepsilon > 0$ as well.

4. Let us characterize the ‘static’ Yang-Mills configurations, i.e. the u -independent solutions to (15), following [9, 10]. Any flat connection \mathcal{A}_{H^3} on H^3 is formally pure gauge,

$$\mathcal{A}_{H^3} = g^{-1} \hat{d}g \quad \text{with} \quad \hat{d} = dy^a \frac{\partial}{\partial y^a}, \quad (18)$$

where \hat{d} is the exterior derivative on H^3 and $g = g(y^a)$ is a smooth map from H^3 into the gauge group G . Since $\partial H^3 = S_\infty^2$ is not empty, the group of admissible gauge transformations is

$$\mathcal{G} = \{g \in C^\infty(H^3, G) \mid g|_{\partial H^3} = \text{Id}\} . \quad (19)$$

The boundary condition on g obstructs the removal of

$$\mathcal{A}_{\partial H^3} = g^{-1} \hat{d}g|_{S_\infty^2} \quad (20)$$

by a gauge transformation and renders the flat connection (18) non-trivial. Hence, the solution space of the equation $\mathcal{F}_{H^3} = 0$ is the infinite-dimensional group

$$\mathcal{N} = C^\infty(H^3, G) , \quad (21)$$

and the moduli space is the quotient group

$$\mathcal{M} = \mathcal{N}/\mathcal{G} = C^\infty(S_\infty^2, G) . \quad (22)$$

The current groups (19), (21) and (22) as well as the corresponding moduli spaces of flat connections are well studied in the literature, see e.g. [9, 10, 11] and references therein. In fact, (19) and (21) are groups of gauge transformations of bundles with and without framing over the boundary S^2 [12], respectively, and the moduli space (22) is their quotient. In our case, the framing (19) is equivalent to imposing the Dirichlet boundary conditions, which are natural for Yang-Mills theory on manifolds with boundary [12].

Notice that the group (22) contains as subgroups the loop group $C^\infty(S^1, G)$ as well as finite-dimensional submanifolds of finite-degree holomorphic maps from $\mathbb{C}P^1 \cong S^2$ into Kähler coset spaces $G/H \subset G$. For a physical interpretation of the moduli space (22), we remark that the gauge equivalence classes of flat connections on H^3 are neither solitonic nor instantonic in character, but rather describe different static Yang-Mills vacua. Here, the term ‘static’ refers to our choice of Lorentz-invariant eigentime τ .² Since $\pi_2(G) = 0$ for any compact connected finite-dimensional group G , the moduli space \mathcal{M} has just a single component.

5. We introduce local coordinates ϕ^α with $\alpha = 1, 2, \dots$ on the moduli space $\mathcal{M} = C^\infty(S_\infty^2, G)$ and assume, following Manton, that \mathcal{A} on the cylinder $\mathbb{R} \times H^3$ given by (11) depends on u (and hence on τ) only via the moduli $\phi^\alpha(u)$. In other words, $\mathcal{A}_{H^3} = g^{-1} \hat{d}g(\phi^\alpha(u); y^a)$, $g(\phi^\alpha(u); \chi \rightarrow \infty)$ is determined by $\phi^\alpha(u)$ and $\mathcal{A}_0(\phi^\alpha(u))$ will be fixed in a moment. This defines a map

$$\phi : \mathbb{R} \rightarrow \mathcal{M} \quad \text{with} \quad \phi(u) = \{\phi^\alpha(u)\} . \quad (23)$$

This map is not free – it is constrained by (16) and (17). Since \mathcal{A}_{H^3} belongs to the solution space \mathcal{N} of flatness equations for any $u \in \mathbb{R}$, its derivative $\partial_0 \mathcal{A}_{H^3}$ is a solution of the flatness condition linearized around \mathcal{A}_{H^3} , i.e. $\partial_0 \mathcal{A}_{H^3}$ belongs to the tangent space $T_{\mathcal{A}} \mathcal{N}$. With the help of the projection $\pi : \mathcal{N} \rightarrow \mathcal{M}$, one can decompose $\partial_0 \mathcal{A}_a$ into two parts,

$$T_{\mathcal{A}} \mathcal{N} = \pi^* T_{\mathcal{A}} \mathcal{M} \oplus T_{\mathcal{A}} \mathcal{G} \quad \Leftrightarrow \quad \partial_0 \mathcal{A}_a = (\partial_0 \phi^\alpha) \xi_{\alpha a} + D_a \epsilon_0 , \quad (24)$$

²For any adiabatic approximation one must pick some temporal foliation.

where $\{\xi_\alpha = \xi_{\alpha a} dy^a\}$ is a local basis of vector fields on \mathcal{M} , and ϵ_0 is a \mathfrak{g} -valued gauge parameter which is determined by the gauge-fixing equation

$$g^{ab} D_a \xi_{\alpha b} = 0 \quad \Leftrightarrow \quad g^{ab} D_a \partial_0 \mathcal{A}_b = g^{ab} D_a D_b \epsilon_0 . \quad (25)$$

Let us fix the gauge on $\mathbb{R} \times H^3$ by choosing $\mathcal{A}_0 = \epsilon_0$. Then (24)–(25) imply that

$$\mathcal{F}_{0b} = \partial_0 \mathcal{A}_b - D_b \mathcal{A}_0 = \partial_0 \mathcal{A}_b - D_b \epsilon_0 = \dot{\phi}^\alpha \xi_{\alpha b} = \pi_* \partial_0 \mathcal{A}_b , \quad (26)$$

where the dot denotes the derivative with respect to $y^0 = u$. From (24)–(26) we then see that (16) is satisfied. Furthermore, we obtain

$$\partial_0 \mathcal{A}_a = \dot{\phi}^\alpha \frac{\partial \mathcal{A}_a}{\partial \phi^\alpha} \quad \Rightarrow \quad \mathcal{A}_0 = \epsilon_0 = \dot{\phi}^\alpha \epsilon_\alpha , \quad (27)$$

where the gauge parameters ϵ_α can be found as solutions to

$$g^{ab} D_a D_b \epsilon_0 = g^{ab} D_a \frac{\partial \mathcal{A}_b}{\partial \phi^\alpha} . \quad (28)$$

6. Substituting (26) into the remaining equation (17), we arrive at

$$g^{ab} \frac{d}{du} (\dot{\phi}^\beta \xi_{\beta b}) = g^{ab} \dot{\phi}^\beta [\xi_{\beta b}, \epsilon_0] . \quad (29)$$

Let us multiply this equation with $\dot{\phi}^\alpha \xi_{\alpha a}$, apply Tr and integrate over H^3 . This yields³

$$\frac{d}{du} (G_{\alpha\beta} \dot{\phi}^\alpha \dot{\phi}^\beta) = 0 , \quad (30)$$

where $G_{\alpha\beta}$ are the metric components on the moduli space \mathcal{M} , defined as

$$G_{\alpha\beta} = - \int_{H^3} d\text{vol} \, g^{ab} \text{Tr}(\xi_{\alpha a} \xi_{\beta b}) . \quad (31)$$

This metric is the standard left-invariant metric on the Lie group. One can get it by left translations from the Killing-Cartan metric on the tangent space at the identity in $\mathcal{M} = C^\infty(S_\infty^2, G)$. However, its calculation may not be an easy task. We postpone its study to the future.

Identifying $y^0 = u$ with the length parameter on \mathcal{M} , i.e. choosing the metric as

$$du^2 = G_{\alpha\beta} d\phi^\alpha d\phi^\beta , \quad (32)$$

(30) becomes the geodesic equation on \mathcal{M} with affine parameter u . To see them in more standard form, consider the action

$$\tilde{S} = \int du \sqrt{G_{\alpha\beta} \dot{\phi}^\alpha \dot{\phi}^\beta} , \quad (33)$$

whose Euler-Lagrange equations are

$$\ddot{\phi}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{\phi}^\beta \dot{\phi}^\gamma - \dot{\phi}^\alpha \frac{d}{du} \ln(G_{\beta\gamma} \dot{\phi}^\beta \dot{\phi}^\gamma) = 0 \quad \stackrel{(30)}{\implies} \quad \ddot{\phi}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{\phi}^\beta \dot{\phi}^\gamma = 0 , \quad (34)$$

³The right-hand side of (29) disappears since $g^{ab} \dot{\phi}^\alpha \dot{\phi}^\beta \text{Tr}([\xi_{\alpha a}, \xi_{\beta b}] \epsilon_0) \equiv 0$.

where the Christoffel symbols are

$$\Gamma_{\beta\gamma}^{\alpha} = \frac{1}{2} G^{\alpha\lambda} \left(\frac{\partial}{\partial\phi^{\gamma}} G_{\beta\lambda} + \frac{\partial}{\partial\phi^{\beta}} G_{\gamma\lambda} - \frac{\partial}{\partial\phi^{\lambda}} G_{\beta\gamma} \right). \quad (35)$$

This derivation reflects the equivalence of the action (33) and the functional

$$S = \int du G_{\alpha\beta} \dot{\phi}^{\alpha} \dot{\phi}^{\beta}. \quad (36)$$

The latter is the effective Yang-Mills action in the adiabatic limit $\varepsilon \rightarrow 0$ and stems from the term

$$\int_{\mathbb{R} \times H^3} d\text{vol} \text{Tr}(\mathcal{F}_{0a} \mathcal{F}^{0a}) \quad (37)$$

in the original Yang-Mills action functional. Since \mathcal{M} is a Lie group, one can construct geodesics as one-parameter subgroups, and we intend to do this in a separate publication. The physical meaning of the moduli parameters ϕ^{α} will become clear from the properties of such solutions.

If we *assume* that $\mathcal{F}_{H^3} = 0$ for any $\tau = e^u$, then (16)–(17) form all Yang-Mills equations on $\mathbb{R} \times H^3$ for any $\varepsilon \neq 0$ including $\varepsilon = 1$.⁴ Their solutions

$$(\mathcal{A}_0, \mathcal{A}_a) = (\dot{\phi}^{\alpha} \epsilon_{\alpha}, g^{-1} \partial_a g(\phi^{\alpha}; \chi, \theta, \varphi)) \quad \text{with} \quad \phi = \phi(u) \quad (38)$$

carry electrical but no magnetic charge since $\mathcal{F}_{0a} \neq 0$ while $\mathcal{F}_{ab} = 0$. From the implicit function theorem it follows that for any solution $\mathcal{A}_{\mu}^{\varepsilon=0}$ defined by ϕ satisfying (34) there exist nearby solutions $\mathcal{A}_{\mu}^{\varepsilon>0}$ of the Yang-Mills equations for ε sufficiently small, and we conjecture that the moduli space of all geodesics (34) in $C^{\infty}(S_{\infty}^2, G)$ is bijective to the moduli space of solutions to the Yang-Mills equations.

7. In conclusion, we reduced Yang-Mills theory on Minkowski space in a certain adiabatic limit to a one-dimensional sigma model with the target space $\mathcal{M} = C^{\infty}(S_{\infty}^2, G)$, which should capture the low-energy dynamics of the gauge theory. We note that the group $C^{\infty}(\Sigma, G)$ of smooth maps from a Riemannian surface Σ (including the case of S^2) into a Lie group G has been considered by mathematicians (see e.g. [13, 14]) but did not yet find a true application in physics. This short article indicates relations of such groups with Yang-Mills theory in four dimensions.

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⁴In general $\mathcal{F}_{H^3} = 0$ is mandatory unless $\varepsilon \rightarrow 0$.

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